Green's Function Treatment of Edge Singularities in the Quasi-TEM Analysis of Microstrip

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Abstract —**A new Green's function approach to the quasi-TEM analysis of microstrip is presented. By expressing the charge density on the strip conductor as the sum of a singular term, derived from the consideration of a Motz expansion, and a continuous term, the integral equation defining this charge density is transformed into an integral equation for the continuous term. An accurate numerical solution to this new integral equation can be obtained by approximating the continuous term by a low-order unit-pulse expansion. It is seen that the numerical scheme developed in this work is both easy to implement and rapidly convergent, thus making it an excellent choice for use in microwave CAD packages.**

I. Introduction

UNDER THE ASSUMPTIONS that loss is negligible and that the mode of propagation is quasi-TEM, the characteristic impedance Z_0 of microstrip is given by [1]

$$Z_0 = \frac{1}{c(CC_a)^{1/2}} \tag{1}$$

where c is the velocity of light in free space, C is the electrostatic capacitance per unit length of microstrip, and C_a is the electrostatic capacitance per unit length of the structure obtained from microstrip by replacing the dielectric substrate with air. A wide variety of methods have been used for the numerical evaluation of C and C_a for both open and shielded microstrip (see, for example, [1]–[8] and the references cited therein). If sufficient computing resources are available, any of the methods cited above can be used to generate accurate design tables or design curves which relate Z_0 to the substrate dielectric constant and line dimensions. In view of the current interest in the computer-aided design (CAD) of microwave integrated circuits (both hybrid and monolithic) [9], attention is focusing on numerical methods which are not only capable of the accurate one-off calculation of Z_0 but are also well suited for use as part of a fast and flexible microwave CAD package. Green's function techniques have been successfully used in microstrip analysis programs [10] and thus appear to be good choices for use in general-purpose microwave CAD packages.

In the mathematical modeling of microstrip, the strip conductor is often taken to be of zero thickness. It is well known from the study of Motz expansions [11] that at the edges of such a mathematically idealized strip both the charge density and the electric field are singular. In the application of numerical techniques to problems involving edge singularities, special consideration of the singularities can often greatly speed the rate of convergence [2], [3], [5], [6], [12], [13]. By employing an analytic approximation for the charge density near the edge of the strip conductor, a new Green's function approach to the numerical analysis of microstrip is developed in this paper. It is seen that this new approach is both rapidly convergent and easy to implement, thus making it an excellent choice for use in microwave CAD packages.

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II. The Green's Function Technique

For an open microstrip of zero thickness (see Fig. 1), the capacitance C can be expressed as

$$C = \int_{-w/2}^{w/2} \sigma(x)\, dx \tag{2}$$

where the charge density $\sigma(x)$ is the solution of the following Fredholm integral equation of the first kind:

$$\int_{-w/2}^{w/2} G(x-x')\sigma(x')\, dx' = 1, \qquad \frac{-w}{2} \leqslant x \leqslant \frac{w}{2}. \tag{3}$$

Here, the kernel $G(x-x')$ is given by [14]

$$G(x-x') = \frac{(1+K)}{4\pi\epsilon_0} \cdot \sum_{n=1}^{\infty} K^{n-1} \ln\left[\frac{(x-x')^2+4n^2h^2}{(x-x')^2+4(n-1)^2h^2}\right] \tag{4}$$

where K, referred to as the partial image coefficient, is given by

$$K = \frac{1-\epsilon_r}{1+\epsilon_r}. \tag{5}$$

By exploiting the symmetry of the microstrip structure about the line $x=0$ ($\sigma(x)=\sigma(-x)$), eq. (3) reduces to

$$\int_0^{w/2} (G(x-x')+G(x+x'))\sigma(x')\, dx' = 1, \qquad 0 \leqslant x \leqslant \frac{w}{2}. \tag{6}$$

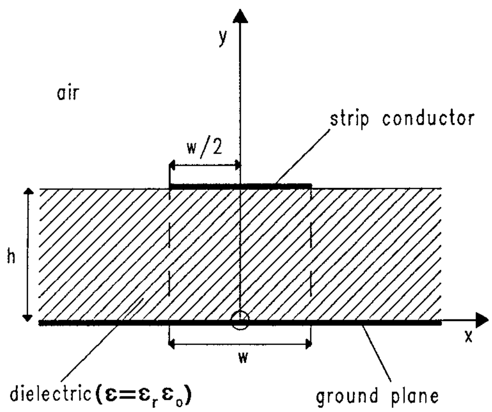

Fig. 1. Cross section of open microstrip.

The capacitance C_a can also be defined via the integral appearing on the right-hand side of (2). However, in this case, the charge density $\sigma(x)$ is the solution of the Fredholm integral equation of the first kind obtained from (3) (or (6)) by replacing the kernel $G(x-x')$ with the kernel $G_a(x-x')$, given by

$$G_a(x-x') = \frac{-1}{4\pi\epsilon_0}\left\{2\ln|x-x'|-\ln\left[(x-x')^2+4h^2\right]\right\}. \tag{7}$$

G_a is simply the Green's function for the classical image problem.

The ideas to be discussed in the rest of this paper hold true for the calculation of both C and C_a. Thus, only the calculation of C need be considered explicitly. General methods for the numerical solution of integral equations of the type described above are well known (see, for example, Lean *et al.* [15]). In essence, these methods involve the approximation of $\sigma(x)$ by a finite sum of the form

$$\sigma(x) \approx \sum_{i=1}^{N} a_i u_i(x) \tag{8}$$

where $u_i(x)$ $(i=1,\cdots,N)$ is a set of known orthogonal expansion functions and a_i $(i=1,\cdots,N)$ is a set of unknown coefficients which are to be determined. The a_i $(i=1,\cdots,N)$ values are determined by first replacing $\sigma(x)$ in the integral equation (3) (or (6)) by the sum given in (8). The integral equation is then reduced, using some standard method such as method of moments or Rayleigh-Ritz, to a system of N linear simultaneous equations for a_i $(i=1,\cdots,N)$. This system of equations can be easily solved using standard methods (e.g., Crout factorization [16]). Once the a_i $(i=1,\cdots,N)$ values are known, an approximation to $\sigma(x)$ can be obtained from (8).

Taking $u_i(x)$ $(i=1,\cdots,N)$ to be a set of unit pulses gives rise to a piecewise constant approximation to $\sigma(x)$. This widely used approximation [14], [17], [18] is frequently referred to as the substrip approximation. If the substrip approximation is used in conjunction with the collocation method, then the elements of the matrix representing the integral operator appearing in (3) (or (6)) can be evaluated analytically and easily coded into a computer program. The charge density on the strip conductor exhibits the following behavior [11]

$$\sigma(x) \sim \sum_{n=0}^{\infty} c_n\left(\frac{w}{2}-|x|\right)^{(2n-1)/2} \quad \text{as } |x| \to \frac{w}{2}- \tag{9}$$

where the c_n $(n=0,\cdots)$ values are constants. The first two terms of the above series both vary rapidly as $|x| \to w/2-$; thus, $\sigma(x)$ cannot be well modeled by a low-order unit-pulse expansion. This explains why the substrip approximation converges only slowly with increasing N.

In order to accurately model the singular behavior of the charge density near the edge of the strip conductor, Silvester and Benedek [12] made the following choice for $u_i(x)$:

$$u_i(x) = \frac{f_i(2x/w)}{\left(\left(\frac{w}{2}\right)^2 - x^2\right)^{1/2}} \tag{10}$$

where

$$f_i(x) = \prod_{n=1}^{i-1}\left[\left(\frac{n}{i-1}\right)^2 - x^2\right], \qquad i>1$$
$$f_1(x) = 1. \tag{11}$$

By employing the method of moments with even-order Legendre polynomials as weighting functions, they were able to calculate reasonably accurate C values using only a two-term series approximation for $\sigma(x)$ (i.e., $N=2$). Similar in spirit to the work of Silvester and Benedek is the work of Gladwell and Coen [13]. The latter applied Galerkin's method using the following expansion functions

$$u_i(x) = \frac{T_{2i}(2x/w)}{\left(\left(\frac{w}{2}\right)^2 - x^2\right)^{1/2}}, \qquad i=0,1,\cdots \tag{12}$$

where T_{2i} is the Chebyshev polynomial of the $2i$th order. They were above to calculate accurate C values using low-order series approximations ($\leqslant$ six terms) for $\sigma(x)$. In the approach of Silvester and Benedek and in the approach of Gladwell and Coen, the evaluation of the elements of the matrix representing the integral operator appearing in (3) involves numerical quadrature and is far more complicated than when using the substrip approximation in conjunction with the collocation method.

III. A New Green's Function Treatment of Microstrip Edge Singularities

In his consideration of the Fredholm integral equation of the first kind defining the charge density on an infinitely long parallel-strip capacitor, Lean [11] wrote the charge density on the capacitor strips as the sum of a singular term, which corresponded to the first term of the series given in (9), and a continuous term. He approximated the continuous term by a polynomial expansion and employed the Rayleigh-Ritz method to determine the coefficient of the singular term, i.e., c_0, and the polynomial expansion coefficients. He was able to calculate very accurate capacitances using very low order polynomial expansions. An approach similar in spirit to Lean's will here be applied to (6).

Consider the function $\lambda(x)$ obtained from $\sigma(x)$ by subtracting the first two terms of the series given in (9), i.e.,

$$\lambda(x)=\sigma(x)-c_0\left(\frac{w}{2}-x\right)^{-1/2}-c_1\left(\frac{w}{2}-x\right)^{1/2},\qquad 0\leqslant x\leqslant\frac{w}{2}. \quad (13)$$

Although $\sigma(x)$ cannot be well modeled by a low-order unit-pulse expansion, it is hoped that $\lambda(x)$ can be. From (6) and (13), it follows that λ satisfies the integral equation

$$\int_0^{w/2}(G(x-x')+G(x+x'))\lambda(x')\,dx' = 1-c_0I_0(x)-c_1I_1(x),\qquad 0\leqslant x\leqslant\frac{w}{2} \quad (14)$$

where

$$I_m(x)=\int_0^{w/2}(G(x-x')+G(x+x'))\cdot\left(\frac{w}{2}-x'\right)^{(2m-1)/2}dx',\qquad m=0,1. \quad (15)$$

From (4), it follows that

$$I_m(x)=\frac{(1+K)}{4\pi\epsilon_0}\sum_{n=1}^{\infty}K^{n-1}(I_{mn}(x)-I_{mn-1}(x)),\qquad m=0,1 \quad (16)$$

where

$$I_{mn}(x)=\int_0^{w/2}\left(\ln\left[(x-x')^2+4n^2h^2\right]+\ln\left[(x+x')^2+4n^2h^2\right]\right)\left(\frac{w}{2}-x'\right)^{(2m-1)/2}dx',\qquad m=0,1;\ n=0,1,\cdots. \quad (17)$$

The integrals I_{mn} $(m=0,1;\ n=0,1,\cdots)$ can all be evaluated analytically and easily coded into a computer program.

Denote by $\lambda_1,\cdots,\lambda_N$ the expansion coefficients of an Nth-order unit-pulse expansion for $\lambda(x)$. By substituting this expansion into (14) and then applying the collocation method, (14) can be reduced to a system of N equations in $N+2$ unknowns, these being λ_i $(i=1,\cdots,N)$ and the Motz expansion coefficients c_0 and c_1. This system of equations can be written in the following matrix form:

$$\sum_{j=1}^{N}G_{ij}\lambda_j=1-c_0I_0(x_i)-c_1I_1(x_i),\qquad i=1,\cdots,N. \quad (18)$$

Here, x_i is the midpoint of the ith unit pulse and

$$G_{ij}=\int_{x_j-w_j/2}^{x_j+w_j/2}(G(x_i-x')+G(x_i+x'))\,dx' \quad (19)$$

where w_j is the width of the jth unit pulse. Apart from the two terms on the right-hand side involving I_0 and I_1, (18) is the same as the matrix equation which would result if the substrip approximation were applied in conjunction with the collocation method to (6). If it is assumed that over the region covered by the two unit pulses nearest the edge of the strip conductor the charge density is accurately represented by the first two terms of (9), then the expansion coefficients corresponding to these two unit pulses, say λ_N and λ_{N-1}, can be set to zero. Thus, the number of unknowns is reduced to N. The system of equations obtained from (18) by setting λ_N and λ_{N-1} to zero can be written in the following form:

$$\sum_{j=1}^{N}A_{ij}b_j=1,\qquad i=1,\cdots,N \quad (20)$$

where

$$b_j=\lambda_j,\qquad j=1,\cdots,N-2$$
$$b_{N-1}=c_0$$
$$b_N=c_1 \quad (21)$$

and

$$A_{ij}=G_{ij},\qquad i=1,\cdots,N;\ j=1,\cdots,N-2$$
$$A_{iN-1}=I_0(x_i),\qquad i=1,\cdots,N$$
$$A_{iN}=I_1(x_i),\qquad i=1,\cdots,N. \quad (22)$$

It should be noted that all of the elements of the matrix A_{ij} $(i=1,\cdots,N;\ j=1,\cdots,N)$ can be evaluated analytically and easily coded into a computer program. By solving (20), it is possible to determine c_0, c_1 and λ_i $(i=1,\cdots,N-2)$. Once these are known, the capacitance C can be calculated as follows:

$$C=2\int_0^{w/2}\sigma(x)\,dx=2\left\{c_0\int_0^{w/2}\left(\frac{w}{2}-x\right)^{-1/2}dx+c_1\int_0^{w/2}\left(\frac{w}{2}-x\right)^{1/2}dx+\sum_{i=1}^{N-2}\lambda_iw_i\right\}$$
$$=4c_0\left(\frac{w}{2}\right)^{1/2}+\frac{4}{3}c_1\left(\frac{w}{2}\right)^{3/2}+2\sum_{i=1}^{N-2}\lambda_iw_i. \quad (23)$$

As an illustrative example, consider the application of the method just described to a microstrip for which $w/h=3$ and $\epsilon_r=10$. Table I shows calculated values for C, C_a, and Z_0 obtained using a selection of N values. For each N, the unit pulses used as expansion functions were all of equal width (i.e., $w_i=w/2N$, $i=1,\cdots,N$); this corresponds to a uniform discretization of the strip conductor. For purposes of comparison, Table I also shows calculated values for C, C_a, and Z_0 obtained by applying the substrip approximation in conjunction with the collocation method to equation (6). (This is referred to as the standard substrip method in Table I.) From Table I, it is clear that the method proposed in this paper is rapidly convergent. Even for $N=2$, which corresponds to approximating the charge density on the strip conductor by the first two terms of the series given in (9), the characteristic impedance Z_0 is correct to four figures. Comparable accuracy cannot be achieved with the standard substrip method, even with $N=480$. Table II shows calculated values for C, C_a, Z_0 obtained using the nonuniform discretization scheme proposed by Atsuki and Yamashita [19]. In this scheme, the widths of the unit pulses used as expansion functions are

TABLE I
CALCULATED MICROSTRIP ($w/h = 3$, $\epsilon_r = 10$) PARAMETERS: UNIFORM DISCRETIZATION

	Standard substrip method			New method		
N	C(pFm^{-1})	C_a(pFm^{-1})	Z_0(ohm)	C(pFm^{-1})	C_a(pFm^{-1})	Z_0(ohm)
2	338.086	45.3910	26.9266	358.830	47.7871	25.4730
5	349.816	46.7806	26.0751	358.738	47.8001	25.4728
10	354.169	47.2816	25.7768	358.757	47.8019	25.4716
20	356.434	47.5394	25.6249	358.761	47.8022	25.4714
40	357.590	47.6702	25.5484	358.761	47.8023	25.4714
80	358.174	47.7361	25.5099	358.761	47.8023	25.4714
160	358.467	47.7691	25.4907	358.761	47.8023	25.4714
320	358.614	47.7857	25.4810	358.761	47.8023	25.4714
480	358.663	47.7912	25.4778	358.761	47.8023	25.4714

TABLE II
CALCULATED MICROSTRIP ($w/h = 3$, $\epsilon_r = 10$) PARAMETERS: NONUNIFORM DISCRETIZATION

	Standard substrip method			New method		
N	C(pFm^{-1})	C_a(pFm^{-1})	Z_0(ohm)	C(pFm^{-1})	C_a(pFm^{-1})	Z_0(ohm)
2	345.975	46.3246	26.3482	358.823	47.7943	25.4713
5	356.355	47.5291	25.6306	358.764	47.8020	25.4713
10	358.130	47.7310	25.5128	358.762	47.8022	25.4714
20	358.600	47.7841	25.4819	358.761	47.8023	25.4714
40	358.720	47.7977	25.4740	358.761	47.8023	25.4714
80	358.751	47.8011	25.4720	358.761	47.8023	25.4714
160	358.759	47.8020	25.4715	358.761	47.8023	25.4714
320	358.761	47.8022	25.4714	358.761	47.8023	25.4714
480	358.761	47.8022	25.4714	358.761	47.8023	25.4714

given by

$$w_i = \frac{w}{2}\left(\sin\left(\frac{i\pi}{2N}\right) - \sin\left(\frac{(i-1)\pi}{2N}\right)\right), \qquad i = 1, \cdots, N. \tag{24}$$

The rate of convergence of the standard substrip method is far greater when using the above nonuniform discretization scheme than when using the uniform discretization scheme. The accuracy obtained with $N = 40$ using the nonuniform discretization scheme is greater than the accuracy obtained with $N = 480$ using the uniform discretization scheme. The rate of convergence of the method proposed in this paper, which is already very good using the uniform discretization scheme, is also improved by using the nonuniform discretization scheme given by (24) and, as can be seen from Table II, is far greater than the rate of convergence of the standard substrip method using the discretization scheme given by (24). The observations made above are consistent with the conclusion made by others [2], [5], [6], [12], [13] that accurate modeling of microstrip edge singularities can lead to substantial reductions in computational storage and time requirements.

IV. CONCLUSIONS

In this paper, a new Green's function approach, incorporating an accurate treatment of edge singularities, to the quasi-TEM analysis of open microstrip has been presented. It is seen that this approach is both easy to implement and rapidly convergent, thus making it an excellent choice for use in microwave CAD packages. The ideas developed in this work can be easily extended to the modeling of coupled microstrip, coplanar strips, and covered versions of microstrip, coupled microstrip, and coplanar strips.

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